

Graphene in external fields

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(Dated: June 22, 2010)

A general discussion of graphene in external electromagnetic field is provided. In general, the formulation is not Lorentz invariant because of Zeeman energy. But it can be restored approximately in the case of strong magnetic field, the condition when quantum Hall effect is observed. Besides obtaining the well-known Hall conductance $\frac{4e^2}{h}(L + 1/2)$, we also provide an explanation of the newly observed Hall conductance $\frac{4e^2}{h}L$ for $L = 0, 1$. These are part of the sequence of Hall conductance $\frac{4e^2}{h}L$ which depends on the filling of Zeeman levels. The energy levels are obtained for general orthogonal constant electric and magnetic field. The second order Dirac equation is derived in applied monochromatic electromagnetic wave, and the major difference between graphene system and conventional Dirac electrons is pointed out.

PACS numbers: 73.43.-f, 72.10.Bg, 72.90.-y, 73.50.-h

Keywords: Graphene, external fields, SO(1,2) invariance, quantum Hall effect

I. INTRODUCTION

Graphene has been one of the major foci in physics because of its simple lattice structure and linear dispersion relation near the Fermi level [1]-[3] when only nearest-neighbor hopping is taken into account. It has become a new testbed not only for condensed matter physics, but also for quantum field theory and mathematical physics [4]-[5]. The physical properties of graphene in external field such as quantum Hall effect (both integer and fractional [6]-[11]), spin quantum Hall effect [12], transport theory [13][14], superconducting [15], and magnetic confinement [16] are under intensive study. Effects of next-nearest-neighbor hopping have also been studied [17]. It is widely recognized that the integer quantum Hall conductance is $\sigma_{xy} = \frac{4e^2}{h}(n + \frac{1}{2})$ where $n = 0, 1, \dots$ [1]. Although disorder and 4-fold symmetry breaking may be utilized to explain the newly found quantum Hall structures $\nu = 0, \pm 1, \pm 4$ [18]-[22], a unified explanation is still called for and the simple structure of $\sigma_{xy} = \frac{4e^2}{h}(n + \frac{1}{2})$ still deserves a simple and fundamental explanation. Not surprisingly, quantum Hall plateaus $\nu = \pm 2, \pm 6, \pm 10$ in graphenes can be explained by using the Landau levels of spinless particle in external magnetic field and the 1+2 Lorentz invariance of the massless Dirac Hamiltonian [23]. Because simple utilisation of Landau levels does not consider the Zeeman energy, which is not negligible compared with low-lying Landau levels, Zeeman energy might explain some of the newly found plateaus. Yet, Zeeman energy is not Lorentz invariant. Therefore, how to use the well-known Landau levels and the Lorentz transformation property of Dirac Hamiltonian remains an issue. In this paper, we discuss graphene in general external fields. Because of the Zeeman energy, the system is not SO(1,2) invariant in general. But when the magnetic field is strong enough, the system does enjoy the invariance approximately, which can be utilized to relate the physics of one experimental configuration to that of another. We obtain the energy levels of the system for the case of constant magnetic fields, and for constant electric and magnetic field. Finally, we derive the second-order Dirac equation in monochromatic electromagnetic waves and point out the difference between graphene system and conventional relativistic electrons. The rest of the paper is organized as follows: in Section II, we present the Hamiltonian in general applied field in the nearest-neighbor hopping approximation and discuss condition of SO(1,2) invariance of the corresponding Lagrangian. In Section III, we discuss the energy levels of the Hamiltonian in applied perpendicular magnetic field. We discuss in Section IV the energy levels in applied orthogonal magnetic and electric fields and derive the quantum Hall conductance in the case of strong magnetic field using the approximate SO(1,2) invariance. A discussion of Dirac equation in applied monochromatic electromagnetic wave is also provided using the method of Volkow in this section. The last section is a brief summary.

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II. HAMILTONIAN IN APPLIED FIELD

The direct lattice of graphene is a superposition of two interpenetrated triangular lattices Λ_A, Λ_B . The generators of lattice Λ_A are [24] $\mathbf{a}_1 = \sqrt{3}a(\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $\mathbf{a}_2 = \sqrt{3}a(\frac{1}{2}, -\frac{\sqrt{3}}{2})$, where $a \approx 1.42 \text{ \AA}$ is the carbon-carbon distance. The vectors $\mathbf{s}_1 = a(0, -1)$, $\mathbf{s}_2 = a(\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $\mathbf{s}_3 = a(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ connect each site in the lattice Λ_A to its nearest neighbor sites in the lattice Λ_B . Unlike regular electron spin, the pseudospin in graphene represents the two sublattices and there is no magnetic moment associated hence does not couple directly to magnetic field[6]. The tight binding Hamiltonian can then be written

$$H_0 = -t \sum_{\sigma} \sum_{\mathbf{r} \in \Lambda_A} \sum_{i=1}^3 [A_{\sigma}^{\dagger}(\mathbf{r}) B_{\sigma}(\mathbf{r} + \mathbf{s}_i) + B_{\sigma}^{\dagger}(\mathbf{r} + \mathbf{s}_i) A_{\sigma}(\mathbf{r})] \quad (1)$$

where σ is pseudo-spin index and t is the uniform hopping constant. In the presence of applied magnetic field \mathbf{B} and electric fields $\mathbf{E} = -\nabla\varphi$, Zeeman energy and Coulomb energy should be included.

$$H_Z = \mu_B [\sum_{\mathbf{r} \in \Lambda_A} \mathbf{B} \cdot A^{\dagger}(\mathbf{r}) \boldsymbol{\tau} A(\mathbf{r}) + \sum_{\mathbf{r} \in \Lambda_B} \mathbf{B} \cdot B^{\dagger}(\mathbf{r}) \boldsymbol{\tau} B(\mathbf{r})] \quad (2)$$

$$H_C = q [\sum_{\mathbf{r} \in \Lambda_A} \varphi(\mathbf{r}) A^{\dagger}(\mathbf{r}) A(\mathbf{r}) + \sum_{\mathbf{r} \in \Lambda_B} \varphi(\mathbf{r}) B^{\dagger}(\mathbf{r}) B(\mathbf{r})] \quad (3)$$

, $\mu_B = \frac{|e|\hbar}{2m_e}$ is the Bohr magneton. In momentum space, with $a_{\mathbf{k}} = \frac{1}{\sqrt{N_A}} \sum_{\mathbf{r} \in \Lambda_A} e^{-i\mathbf{k} \cdot \mathbf{r}} A(\mathbf{r})$, $b_{\mathbf{k}} = \frac{1}{\sqrt{N_B}} \sum_{\mathbf{r} \in \Lambda_B} e^{-i\mathbf{k} \cdot \mathbf{r}} B(\mathbf{r})$, where N_A is the number of lattice points of sublattice Λ_A (or Λ_B), $H_0 + H_Z$ reads in \mathbf{k} -space

$$H = -t \sum_{\mathbf{k}} (f_{\mathbf{k}} a_{\sigma\mathbf{k}}^{\dagger} b_{\sigma\mathbf{k}} + f_{\mathbf{k}}^* b_{\sigma\mathbf{k}}^{\dagger} a_{\sigma\mathbf{k}}) + \mu_B \mathbf{B} \cdot \sum_{\mathbf{k}} (a_{\mathbf{k}}^{\dagger} \boldsymbol{\tau} a_{\mathbf{k}} + b_{\mathbf{k}}^{\dagger} \boldsymbol{\tau} b_{\mathbf{k}}) \quad (4)$$

with $f_{\mathbf{k}} = \sum_{i=1}^3 e^{i\mathbf{k} \cdot \mathbf{s}_i}$. $t \approx 3.033 \text{ eV}$.

$$f_{\mathbf{k}} = e^{-ik_y a} + 2e^{\frac{i}{2}k_y a} \cos(\frac{\sqrt{3}}{2}k_x a) \quad (5)$$

The lattice Hamiltonian H_0 vanishes at the six corners of the first Brillouin zone. Among these, only two are inequivalent, and can be chosen as

$$\mathbf{K}_{\pm} = \pm \left(\frac{4\pi}{3\sqrt{3}a}, 0 \right); f_{\mathbf{K}_{\pm}} = 0. \quad (6)$$

Let $\mathbf{k} = \mathbf{K}_{\pm} + \mathbf{p}/\hbar$. In the vicinity of \mathbf{K}_{+} , we have

$$f_{\mathbf{k}} = -\frac{3}{2}p_x a/\hbar - \frac{3}{2}ip_y a/\hbar + \frac{3}{8}a^2(p_x - ip_y)^2/\hbar^2 + \dots \quad (7)$$

In the vicinity of \mathbf{K}_{-} , we have

$$f_{\mathbf{k}} = \frac{3}{2}p_x a/\hbar - \frac{3}{2}ip_y a/\hbar + \frac{3}{8}a^2(p_x + ip_y)^2/\hbar^2 + \dots \quad (8)$$

After defining two-component spinors as $\psi_{\mathbf{k}} = (a_{\mathbf{k}}, b_{\mathbf{k}})^T$, $\psi_{\mathbf{k}}^{\dagger} = (a_{\mathbf{k}}^{\dagger}, b_{\mathbf{k}}^{\dagger})$, $\psi_{\mathbf{k}\sigma 1} = a_{\mathbf{k}\sigma}$, $\psi_{\mathbf{k}\sigma 2} = b_{\mathbf{k}\sigma}$, $\sigma = \pm 1$. we have

$$H = -t \sum_{\mathbf{k}\sigma} \psi_{\mathbf{k}\sigma}^{\dagger} \begin{pmatrix} 0 & f_{\mathbf{k}} \\ f_{\mathbf{k}}^* & 0 \end{pmatrix} \psi_{\mathbf{k}\sigma} + \mu_B \mathbf{B} \cdot \sum_{\mathbf{k}} (\psi_{\mathbf{k}\alpha 1}^{\dagger} \boldsymbol{\sigma}_{\alpha\beta} \psi_{\mathbf{k}\beta 1} + \psi_{\mathbf{k}\alpha 2}^{\dagger} \boldsymbol{\sigma}_{\alpha\beta} \psi_{\mathbf{k}\beta 2}) + H_C \quad (9)$$

When H_0 is linearized around these two points, one obtains

$$H_{0|\mathbf{k}=\mathbf{K}_{+}+\mathbf{p}} = v_F \begin{pmatrix} 0 & p_x + ip_y \\ p_x - ip_y & 0 \end{pmatrix} = v_F \boldsymbol{\alpha} \cdot \mathbf{p} \quad (10)$$

where $v_F = \frac{3ta}{2\hbar}$, $\alpha = (\sigma_x, -\sigma_y)$. and

$$H_0|_{\mathbf{k}=\mathbf{K}_-+\mathbf{p}} = -v_F \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix} = -v_F \boldsymbol{\sigma} \cdot \mathbf{p} \quad (11)$$

$\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$. In the case $\mathbf{B} = (0, 0, B)$, we have

$$H = -t \sum_{\mathbf{k}\sigma} \psi_{\mathbf{k}\sigma}^\dagger \begin{pmatrix} 0 & f_{\mathbf{k}} \\ f_{\mathbf{k}}^* & 0 \end{pmatrix} \psi_{\mathbf{k}\sigma} + \mu_B B \sum_{\mathbf{k}} (\psi_{\mathbf{k}+}^\dagger \psi_{\mathbf{k}+} - \psi_{\mathbf{k}-}^\dagger \psi_{\mathbf{k}-}) + H_C \quad (12)$$

Or, let $\psi_{\mathbf{k}+} = \xi_{\mathbf{k}} = \begin{pmatrix} a_{\mathbf{k}+} \\ b_{\mathbf{k}+} \end{pmatrix}$, $\psi_{\mathbf{k}-} = \eta_{\mathbf{k}} = \begin{pmatrix} a_{\mathbf{k}-} \\ b_{\mathbf{k}-} \end{pmatrix}$, we have

$$H = -t \sum_{\mathbf{k}} \xi_{\mathbf{k}}^\dagger \begin{pmatrix} 0 & f_{\mathbf{k}} \\ f_{\mathbf{k}}^* & 0 \end{pmatrix} \xi_{\mathbf{k}} - t \sum_{\mathbf{k}} \eta_{\mathbf{k}}^\dagger \begin{pmatrix} 0 & f_{\mathbf{k}} \\ f_{\mathbf{k}}^* & 0 \end{pmatrix} \eta_{\mathbf{k}} + \mu_B B \sum_{\mathbf{k}} (\xi_{\mathbf{k}}^\dagger \xi_{\mathbf{k}} - \eta_{\mathbf{k}}^\dagger \eta_{\mathbf{k}}) + H_C \quad (13)$$

Close to \mathbf{K}_+ , we have

$$H = \sum_{\mathbf{p}} \xi_{\mathbf{p}}^\dagger (v_F \boldsymbol{\alpha} \cdot \mathbf{p} + \mu_B B) \xi_{\mathbf{p}} + \sum_{\mathbf{p}} \eta_{\mathbf{p}}^\dagger (v_F \boldsymbol{\alpha} \cdot \mathbf{p} - \mu_B B) \eta_{\mathbf{p}} + H_C \quad (14)$$

There are two inequivalent representations of the γ -matrices in three dimensions: $\gamma^\mu = (\sigma_3, i\sigma_2, i\sigma_1)$ and $\gamma^\mu = (-\sigma_3, -i\sigma_2, -i\sigma_1)$. We choose the first.

$$H = \sum_{\mathbf{p}} \bar{\xi}_{\mathbf{p}} (v_F \boldsymbol{\gamma} \cdot \mathbf{p} + \mu_B B \gamma^0) \xi_{\mathbf{p}} + \sum_{\mathbf{p}} \bar{\eta}_{\mathbf{p}} (v_F \boldsymbol{\gamma} \cdot \mathbf{p} - \mu_B B \gamma^0) \eta_{\mathbf{p}} + H_C \quad (15)$$

where $\bar{\xi} = \xi^\dagger \gamma^0$. Incorporated $U(1)$ gauge invariance, the Hamiltonian reads

$$H = \int d^2 \mathbf{x} \bar{\xi}(x) [\hbar v_F \boldsymbol{\gamma} \cdot (-i\nabla - \frac{q}{\hbar} \mathbf{A}) + \mu_B B \gamma^0 + q \gamma^0 \varphi] \xi(x) + \int d^2 \mathbf{x} \bar{\eta}(x) [\hbar v_F \boldsymbol{\gamma} \cdot (-i\nabla - \frac{q}{\hbar} \mathbf{A}) - \mu_B B \gamma^0 + q \gamma^0 \varphi] \eta(x) \quad (16)$$

Let $x^\mu = (x^0, \mathbf{x}) = (v_F t, x, y)$, $A^\mu = (A^0, \mathbf{A})$. Denoting $D_\mu = \partial_\mu + i \frac{q}{\hbar} A_\mu$, $A^0 = \varphi/v_F$, we have

$$\mathcal{L} = \xi^\dagger i \hbar \partial_t \xi - \mathcal{H} = \bar{\xi}(x) (i \hbar \not{D} + g B \gamma^0) \xi(x) + \bar{\eta}(x) (i \hbar \not{D} - g B \gamma^0) \eta(x) \quad (17)$$

where $g = \mu_B/v_F$. The $U(1)$ gauge invariance is preserved but the Lorentz $SO(1,2)$ invariance is broken by the Zeeman term. Since

$$-\varepsilon^{\mu\nu\tau} \partial_\mu A_\nu \gamma_\tau = B \gamma^0 + \frac{1}{v_F} E_x \gamma^2 - \frac{1}{v_F} E_y \gamma^1 \quad (18)$$

For $|\mathbf{E}| \ll v_F |\mathbf{B}|$, we can write

$$B \gamma^0 \approx -\varepsilon^{\mu\nu\tau} \partial_\mu A_\nu \gamma_\tau \quad (19)$$

and in this case

$$L = \int d^3 x \left[\bar{\xi}(x) (i \hbar \not{D} - g \varepsilon^{\mu\nu\tau} \partial_\mu A_\nu \gamma_\tau) \xi(x) + \bar{\eta}(x) (i \hbar \not{D} + g \varepsilon^{\mu\nu\tau} \partial_\mu A_\nu \gamma_\tau) \eta(x) \right] \quad (20)$$

which shows 1+2 Lorentz invariance. The current is

$$j^\mu = -\frac{\delta L}{\delta A_\mu} = q \bar{\xi} \gamma^\mu \xi + q \bar{\eta} \gamma^\mu \eta + g \varepsilon^{\mu\nu\tau} \partial_\nu (\bar{\xi} \gamma_\tau \xi) - g \varepsilon^{\mu\nu\tau} \partial_\nu (\bar{\eta} \gamma_\tau \eta) \quad (21)$$

III. CONSTANT PERPENDICULAR MAGNETIC FIELD

In this case $\varphi = 0$ and we want to calculate the grand-canonical partition function. $Z = \text{Tr } e^{-\beta K}$, $K = H - \mu N$. The Dirac equation is

$$\left[\hbar v_F \boldsymbol{\alpha} \cdot (-i\nabla - \frac{q}{\hbar} \mathbf{A}) + \mu_B B - \mu \right] \xi = K \xi \quad (22)$$

which turns into second order

$$(-D_i D_i + \frac{q}{\hbar} B \sigma_3) \xi = \frac{(K - \mu_B B + \mu)^2}{\hbar^2 v_F^2} \xi \quad (23)$$

Using standard Landau levels (assuming $qB = |qB|$), we have

$$(K - \mu_B B + \mu)^2 = \hbar v_F^2 2qB(\ell + 1/2 - s_z) \quad (24)$$

So

$$K_{\ell, s_z} = \pm \hbar v_F \sqrt{2 \frac{q}{\hbar} B(\ell + 1/2 - s_z)} + \mu_B B - \mu \quad (25)$$

The partition function is

$$Z = \prod_{\ell, s_z} [1 + e^{-\beta K_{\ell, s_z}}]^{\Delta_L} \quad (26)$$

where $\Delta_L = \frac{|qB|}{2\pi\hbar}$ is the Landau degeneracy per unit area. For $B = 1\text{T}$, we have $\Delta_L \approx 2.4 \times 10^{14}/m^2$.

$$-\Gamma = \Delta_L \sum_{\ell} \sum_{s_z} \ln[1 + e^{-\beta K_{\ell, s_z}}] \quad (27)$$

For $\mu = 0$, we have the energy levels

$$K_{\ell, s_z} = \pm \hbar v_F \sqrt{2 \frac{q}{\hbar} B(\ell + 1/2 - s_z)} + \mu_B B \quad (28)$$

Corresponding to field η , we have

$$K_{\ell, s_z} = \pm \hbar v_F \sqrt{2 \frac{q}{\hbar} B(\ell + 1/2 - s_z)} - \mu_B B \quad (29)$$

So the energy levels are symmetric under $+\leftrightarrow -$. Therefore if the graphene is undoped, the Fermi level is still at $\mu = 0$. Suppose $\mu_B B > 0$. For ξ -field, $K_{\ell, 1/2}$ levels are ($[A]$ represents integer part of A)

$$s_z = 1/2 : \begin{cases} \mu_B B - \hbar|v_F| \sqrt{2 \frac{q}{\hbar} B \ell}, & \ell = 0, 1, \dots, [\frac{\mu_B^2 B}{2q\hbar v_F^2}], \text{ positive} \\ \mu_B B - \hbar|v_F| \sqrt{2 \frac{q}{\hbar} B \ell}, & \ell = [\frac{\mu_B^2 B}{2q\hbar v_F^2}] + 1, \dots, \infty, \text{ negative} \\ \mu_B B + \hbar|v_F| \sqrt{2 \frac{q}{\hbar} B \ell}, & \ell = 1, 2, \dots, \infty, \text{ positive} \end{cases} \quad (30)$$

$K_{\ell, -1/2}$ levels are

$$s_z = -1/2 : \begin{cases} \mu_B B - \hbar|v_F| \sqrt{2 \frac{q}{\hbar} B(\ell + 1)}, & \ell = 0, 1, \dots, [\frac{\mu_B^2 B}{2q\hbar v_F^2}] - 1, \text{ positive} \\ \mu_B B - \hbar|v_F| \sqrt{2 \frac{q}{\hbar} B(\ell + 1)}, & \ell = [\frac{\mu_B^2 B}{2q\hbar v_F^2}], \dots, \infty, \text{ negative} \\ \mu_B B + \hbar|v_F| \sqrt{2 \frac{q}{\hbar} B(\ell + 1)}, & \ell = 0, 1, 2, \dots, \infty, \text{ positive} \end{cases} \quad (31)$$

For η -field, $K_{\ell, 1/2}$ levels are

$$s_z = 1/2 : \begin{cases} -\mu_B B + \hbar|v_F| \sqrt{2 \frac{q}{\hbar} B \ell}, & \ell = 0, 1, \dots, [\frac{\mu_B^2 B}{2q\hbar v_F^2}], \text{ negative} \\ -\mu_B B + \hbar|v_F| \sqrt{2 \frac{q}{\hbar} B \ell}, & \ell = [\frac{\mu_B^2 B}{2q\hbar v_F^2}] + 1, \dots, \infty, \text{ positive} \\ -\mu_B B - \hbar|v_F| \sqrt{2 \frac{q}{\hbar} B \ell}, & \ell = 1, 2, \dots, \infty, \text{ negative} \end{cases} \quad (32)$$

$K_{\ell,-1/2}$ levels are

$$s_z = -1/2 : \begin{cases} -\mu_B B + \hbar|v_F| \sqrt{2\frac{q}{\hbar} B(\ell+1)}, & \ell = 0, 1, \dots, [\frac{\mu_B^2 B}{2q\hbar v_F^2}] - 1, \text{negative} \\ -\mu_B B + \hbar|v_F| \sqrt{2\frac{q}{\hbar} B(\ell+1)}, & \ell = [\frac{\mu_B^2 B}{2q\hbar v_F^2}], \dots, \infty, \text{positive} \\ -\mu_B B - \hbar|v_F| \sqrt{2\frac{q}{\hbar} B(\ell+1)}, & \ell = 0, 1, 2, \dots, \infty, \text{negative} \end{cases} \quad (33)$$

So negative levels comes from both ξ and η fields.

$$\xi : K_{\ell,1/2} : \mu_B B - \hbar|v_F| \sqrt{2\frac{q}{\hbar} B\ell}, \ell = [\frac{\mu_B^2 B}{2q\hbar v_F^2}] + 1, \dots, \infty \quad (34)$$

$$\xi : K_{\ell,-1/2} : \mu_B B - \hbar|v_F| \sqrt{2\frac{q}{\hbar} B(\ell+1)}, \ell = [\frac{\mu_B^2 B}{2q\hbar v_F^2}], \dots, \infty \quad (35)$$

$$\eta : K_{\ell,1/2} : \begin{cases} -\mu_B B + \hbar|v_F| \sqrt{2\frac{q}{\hbar} B\ell}, & \ell = 0, 1, \dots, [\frac{\mu_B^2 B}{2q\hbar v_F^2}] \\ -\mu_B B - \hbar|v_F| \sqrt{2\frac{q}{\hbar} B\ell}, & \ell = 1, 2, \dots, \infty \end{cases} \quad (36)$$

$$\eta : K_{\ell,-1/2} : \begin{cases} -\mu_B B + \hbar|v_F| \sqrt{2\frac{q}{\hbar} B(\ell+1)}, & \ell = 0, 1, \dots, [\frac{\mu_B^2 B}{2q\hbar v_F^2}] - 1 \\ -\mu_B B - \hbar|v_F| \sqrt{2\frac{q}{\hbar} B(\ell+1)}, & \ell = 0, 1, 2, \dots, \infty \end{cases} \quad (37)$$

For $t = 3.033$ eV, $v_F = \frac{3ta}{2\hbar} \approx 10^6$ (m/s). Defining $B_0 := \frac{2qv_F^2 \hbar}{\mu_B^2}$, then $B_0 \approx 1.1 \times 10^6$ T. For lab field $B \sim 10$ T, hence, $[\frac{B}{B_0}] = 0$. Thus for ξ -field, $K_{\ell,1/2}$ levels are

$$s_z = 1/2 : \begin{cases} \mu_B B, & \ell = 0, \text{positive} \\ \mu_B B - \hbar|v_F| \sqrt{2\frac{q}{\hbar} B\ell}, & \ell = 1, \dots, \infty, \text{negative} \\ \mu_B B + \hbar|v_F| \sqrt{2\frac{q}{\hbar} B\ell}, & \ell = 1, 2, \dots, \infty, \text{positive} \end{cases} \quad (38)$$

$K_{\ell,-1/2}$ levels are

$$s_z = -1/2 : \begin{cases} \mu_B B - \hbar|v_F| \sqrt{2\frac{q}{\hbar} B(\ell+1)}, & \ell = 0, \dots, \infty, \text{negative} \\ \mu_B B + \hbar|v_F| \sqrt{2\frac{q}{\hbar} B(\ell+1)}, & \ell = 0, 1, 2, \dots, \infty, \text{positive} \end{cases} \quad (39)$$

For η -field, $K_{\ell,1/2}$ levels are

$$s_z = 1/2 : \begin{cases} -\mu_B B, & \ell = 0, \text{negative} \\ -\mu_B B + \hbar|v_F| \sqrt{2\frac{q}{\hbar} B\ell}, & \ell = 1, \dots, \infty, \text{positive} \\ -\mu_B B - \hbar|v_F| \sqrt{2\frac{q}{\hbar} B\ell}, & \ell = 1, 2, \dots, \infty, \text{negative} \end{cases} \quad (40)$$

$K_{\ell,-1/2}$ levels are

$$s_z = -1/2 : \begin{cases} -\mu_B B + \hbar|v_F| \sqrt{2\frac{q}{\hbar} B(\ell+1)}, & \ell = 0, \dots, \infty, \text{positive} \\ -\mu_B B - \hbar|v_F| \sqrt{2\frac{q}{\hbar} B(\ell+1)}, & \ell = 0, 1, 2, \dots, \infty, \text{negative} \end{cases} \quad (41)$$

Note $s_z = 1/2$ belongs to sublattice Λ_A and $s_z = -1/2$ belongs to sublattice Λ_B . Consider negative levels For ξ -field, $K_{\ell,1/2}$ levels are

$$s_z = 1/2 : \{ \mu_B B - \hbar|v_F| \sqrt{2\frac{q}{\hbar} B\ell}, \ell = 1, \dots, \infty, \text{negative} \} \quad (42)$$

$K_{\ell,-1/2}$ levels are

$$s_z = -1/2 : \{ \mu_B B - \hbar|v_F| \sqrt{2\frac{q}{\hbar} B(\ell+1)}, \ell = 0, \dots, \infty, \text{negative} \} \quad (43)$$

For η -field, $K_{\ell,1/2}$ levels are

$$s_z = 1/2 : \begin{cases} -\mu_B B, & \ell = 0, \text{negative} \\ -\mu_B B - \hbar|v_F|\sqrt{2\frac{q}{\hbar}B\ell}, & \ell = 1, 2, \dots, \infty, \text{negative} \end{cases} \quad (44)$$

$K_{\ell,-1/2}$ levels are

$$s_z = -1/2 : \{ -\mu_B B - \hbar|v_F|\sqrt{2\frac{q}{\hbar}B(\ell+1)}, \ell = 0, 1, 2, \dots, \infty, \text{negative} \} \quad (45)$$

So levels

$$\mu_B B - \hbar|v_F|\sqrt{2\frac{q}{\hbar}B\ell}, \ell = 1, \dots, \infty$$

and levels

$$-\mu_B B - \hbar|v_F|\sqrt{2\frac{q}{\hbar}B\ell}, \ell = 1, 2, \dots, \infty$$

are doubly degenerate (apart from the Landau degeneracy). We can not have an infinite number of negative levels filled-up since a cut-off is necessary. Suppose among the levels $\mu_B B - \hbar|v_F|\sqrt{2\frac{q}{\hbar}B\ell}$, $\ell = L_1^u \dots, L_1^l$ are filled, and among the levels $-\mu_B B - \hbar|v_F|\sqrt{2\frac{q}{\hbar}B\ell}$, $\ell = L_2^u \dots, L_2^l$ are filled. If $-\mu_B B$ is also filled, then at $T = 0K$

$$N = \Delta_L (2(L_1^l - L_1^u + 1) + 2(L_2^l - L_2^u + 1) + 1) = 2\Delta_L (L_1 + L_2 + \frac{1}{2}) \quad (46)$$

where $L_1 = L_1^l - L_1^u + 1$, $L_2 = L_2^l - L_2^u + 1$. If $-\mu_B B$ is not filled, i.e., $\mu < -\mu_B B$, then

$$N = \Delta_L (2(L_1^l - L_1^u + 1) + 2(L_2^l - L_2^u + 1)) = 2\Delta_L (L_1 + L_2) \quad (47)$$

The physics near \mathbf{K}_- makes equal contribution. The magnetization is

$$M = \mu_B (N_\xi - N_\eta) \quad (48)$$

where

$$N_\xi = \sum_{\mathbf{p}} \langle \xi_{\mathbf{p}}^\dagger \xi_{\mathbf{p}} \rangle, \quad N_\eta = \sum_{\mathbf{p}} \langle \eta_{\mathbf{p}}^\dagger \eta_{\mathbf{p}} \rangle, \quad (49)$$

The effective potential for ξ -fields should be regularized. The sum in the second term in the following formula actually extends to only L_1 instead of to infinity. Hence

$$-\frac{\Gamma_\xi}{\Delta_L} = \ln \left[1 + e^{\beta(\mu - \mu_B B)} \right] + 2 \sum_{\ell=1}^{\infty} \ln \left[1 + e^{\beta(\mu - \hbar v_F \sqrt{2\frac{q}{\hbar}B\ell} - \mu_B B)} \right] + 2 \sum_{\ell=1}^{L_1} \ln \left[1 + e^{\beta(\mu + \hbar v_F \sqrt{2\frac{q}{\hbar}B\ell} - \mu_B B)} \right] \quad (50)$$

With $\langle N \rangle = \beta^{-1} \frac{\partial \ln Z}{\partial \mu} = -\beta^{-1} \frac{\partial \Gamma}{\partial \mu}$, we have

$$\langle N \rangle_\xi = \Delta_L \left[\frac{1}{1 + e^{\beta(\mu_B B - \mu)}} + 2 \sum_{\ell=1}^{\infty} \frac{1}{1 + e^{\beta(\hbar v_F \sqrt{2\frac{q}{\hbar}B\ell} - \mu_B B - \mu)}} + 2 \sum_{\ell=1}^{L_1} \frac{1}{1 + e^{\beta(-\hbar v_F \sqrt{2\frac{q}{\hbar}B\ell} + \mu_B B - \mu)}} \right] \quad (51)$$

Similarly

$$\langle N \rangle_\eta = \Delta_L \left[\frac{1}{1 + e^{\beta(-\mu_B B - \mu)}} + 2 \sum_{\ell=1}^{\infty} \frac{1}{1 + e^{\beta(\hbar v_F \sqrt{2\frac{q}{\hbar}B\ell} - \mu_B B - \mu)}} + 2 \sum_{\ell=1}^{L_2} \frac{1}{1 + e^{\beta(-\hbar v_F \sqrt{2\frac{q}{\hbar}B\ell} - \mu_B B - \mu)}} \right] \quad (52)$$

At zero temperature, $\mu = 0$, and we have

$$\langle N \rangle = \langle N \rangle_\xi + \langle N \rangle_\eta = \Delta_L [1 + 2(L_1 + L_2)] \quad (53)$$

and the magnetization

$$M = \Delta_L \mu_B (2L_1 - 2L_2 - 1) \quad (54)$$

Since $L_1 = L_2, L_2 + 1$, we have $M = \pm \Delta_L \mu_B$.

IV. PERPENDICULAR MAGNETIC AND ELECTRIC FIELDS

A. Constant fields: $|\mathbf{E}| < v_F |\mathbf{B}|$

In this case, we can use the 2+1 Lorentz invariance and the results of previous section. Suppose system Σ' is moving at velocity \mathbf{v} relative to system Σ . For $\mathbf{v} = (v, 0)$, the SO(1,2) transformation is

$$A'_\mu = \Lambda_\mu{}^\nu A_\nu \quad (55)$$

$$\Lambda_\mu{}^\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (56)$$

where $\beta = \frac{v}{v_F}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/v_F & E_y/v_F \\ -E_x/v_F & 0 & -B_z \\ -E_y/v_F & B_z & 0 \end{pmatrix} \quad (57)$$

The electromagnetic fields are related by

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel, \quad \mathbf{E}'_\perp = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B})_\perp \quad (58)$$

$$\mathbf{B}'_\parallel = \mathbf{B}_\parallel, \quad \mathbf{B}'_\perp = \gamma(\mathbf{B} - \frac{\mathbf{v}}{v_F^2} \times \mathbf{E})_\perp \quad (59)$$

Suppose we have in Σ : $\mathbf{E} = 0$, $\mathbf{B} = (0, 0, B)$, then in Σ' .

$$\mathbf{E}'_\parallel = 0, \quad \mathbf{E}'_\perp = \gamma \mathbf{v} \times \mathbf{B} \quad (60)$$

$$\mathbf{B}'_\parallel = \mathbf{B}_\parallel, \quad \mathbf{B}'_\perp = \gamma \mathbf{B}_\perp \quad (61)$$

We want $\mathbf{E}' = (0, E'_y, 0)$, $\mathbf{B}' = (0, 0, B')$. Hence we can choose $\mathbf{v} = (v, 0)$. In this case.

$$E'_y = -\gamma v B, \quad B'_z = \gamma B \quad (62)$$

So we have

$$v = -\frac{E'_y}{B'_z} \quad (63)$$

So in Σ

$$j^{0'} = \gamma j^0, j^{x'} = -\gamma \beta j^0 = \gamma \frac{E'_y}{v_F B'_z} j^0 \quad (64)$$

Note that

$$j^\mu = q(v_F \rho, \mathbf{j}) \quad (65)$$

The Hall conductance is

$$\sigma'_{xy} = \frac{\gamma}{v_F B'_z} j^0 = \frac{q^2}{h} (2L + 1) \quad (66)$$

if $-\mu_B B$ is filled or

$$\sigma'_{xy} = \frac{\gamma}{v_F B'_z} j^0 = \frac{q^2}{h} 2L \quad (67)$$

if $-\mu_B B$ is not filled, where $L = L_1 + L_2$. The physics near \mathbf{K}_- makes equal contribution, hence

$$\sigma'_{xy} = \frac{\gamma}{v_F B'_z} j^0 = \begin{cases} \frac{4q^2}{h} (L + \frac{1}{2}), & -\mu_B B \text{ is filled} \\ \frac{4q^2}{h} L, & -\mu_B B \text{ is not filled} \end{cases} \quad (68)$$

The above SO(1,2) transformation breaks down when $v \geq v_F$, which means when $|E'_y| \geq v_F |B'_z|$. The magnetic moment current

$$j_s^\mu = \mu_B (v_F (\rho_\xi - \rho_\eta), \dot{j}_\xi - \dot{j}_\eta) \quad (69)$$

we then have

$$j_s^{x'} = (-\gamma\beta) v_F M = \mu_B \frac{qE'_y}{h} \quad (70)$$

So far the sequence of Hall conductance $\sigma_{xy} = \frac{4q^2}{h} (L + \frac{1}{2})$ has been observed, we here predict the existence of the sequence $\sigma_{xy} = \frac{4q^2}{h} L$ and the recently observed $L = 0, 4$ are just part of this sequence[22]. According to above analysis, the filling of Zeeman levels makes the difference.

B. General orthogonal constant B and E

When the condition $|E| \ll v_F |B|$ does not hold, we need to solve the Dirac equation directly. Let $\mathbf{A} = (-yB, 0)$, $\varphi = yE$. The second order Dirac equation reads

$$\left[-D_i D_i + \frac{q}{h} B \sigma^z + \frac{iqE}{\hbar v_F} \sigma^y - \frac{(qEy - K + \mu_B B)^2}{\hbar^2 v_F^2} \right] \xi = 0 \quad (71)$$

Denoting $\varepsilon = \frac{qE}{\hbar v_F}$, $\kappa = \frac{K - \mu_B B}{\hbar v_F}$,

$$\left[-D_i D_i + 2\pi \Delta_L \sigma^z + i\varepsilon \sigma^y - (\varepsilon y - \kappa)^2 \right] \xi = 0 \quad (72)$$

We can diagonalize the part $2\pi \Delta_L \sigma^z + i\varepsilon \sigma^y$ by Jordan decomposition (a similarity transformation).

$$2\pi \Delta_L \sigma^z + i\varepsilon \sigma^y = S J S^{-1} \quad (73)$$

where

$$S = \begin{pmatrix} \frac{1}{\varepsilon} (-2\pi \Delta_L + \sqrt{4\pi^2 \Delta_L^2 - \varepsilon^2}) & \frac{1}{\varepsilon} (-2\pi \Delta_L - \sqrt{4\pi^2 \Delta_L^2 - \varepsilon^2}) \\ 1 & 1 \end{pmatrix} \quad (74)$$

$$J = \begin{pmatrix} -\sqrt{4\pi^2 \Delta_L^2 - \varepsilon^2} & 0 \\ 0 & \sqrt{4\pi^2 \Delta_L^2 - \varepsilon^2} \end{pmatrix} := \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \quad (75)$$

Let

$$\xi = S \Psi \quad (76)$$

Then

$$\left[-D_i D_i + J - (\varepsilon y - \kappa)^2 \right] \Psi = 0 \quad (77)$$

$$\left[-\partial_y^2 - \partial_x^2 - \frac{2iqB}{h} y \partial_x + \frac{q^2 B^2}{h^2} y^2 + J - (\varepsilon y - \kappa)^2 \right] \Psi = 0 \quad (78)$$

$$\left[-\partial_y^2 - \partial_x^2 - 4\pi \Delta_L y i \partial_x + 4\pi^2 \Delta_L^2 y^2 + J - (\varepsilon y - \kappa)^2 \right] \Psi = 0 \quad (79)$$

Let

$$\Psi = e^{ip_x x/\hbar} f(y) \quad (80)$$

Denoting $k_x = p_x/\hbar$.

$$\left[-\partial_y^2 + k_x^2 + 4\pi\Delta_L y k_x + 4\pi^2\Delta_L^2 y^2 + J - (\varepsilon y - \kappa)^2 \right] f = 0 \quad (81)$$

Let

$$\alpha = 4\pi^2\Delta_L^2 - \varepsilon^2 \quad b = 4\pi\Delta_L k_x + 2\varepsilon\kappa \quad c = k_x^2 - \kappa^2 + J_1 \quad (82)$$

$$(-\partial_y^2 + \alpha y^2 + by + c)f_1 = 0 \quad (83)$$

Let

$$u = y + \frac{b}{2\alpha} \quad (84)$$

then

$$(-\partial_u^2 + \alpha u^2 + c - \frac{b^2}{4\alpha})f_1 = 0 \quad (85)$$

When $\alpha > 0$, this is a harmonic problem while for $\alpha < 0$, J_1 imaginary. Similar issue exists in [25], thereof β can be larger than 1, hence γ is imaginary. The critical value of B is $B_c = E/v_F$. For $B < B_c$, the solution should be parabolic cylinder functions, as in the case of [26].

For positive α , the problem is harmonic oscillator with $m = \hbar^2/2$, $\omega = 2\sqrt{\alpha}/\hbar$. Hence we have

$$b^2 - 4\alpha c = 4\alpha\hbar\omega(n + 1/2) \quad (86)$$

We have

$$\kappa = \frac{-\varepsilon k_x \pm \sqrt{\alpha J_{1,2} + \alpha^{3/2} + 2\alpha^{3/2}n}}{2\pi\Delta_L} \quad (87)$$

For f_2 , $J_2 = \sqrt{\alpha}$, we have

$$\kappa = \frac{-\varepsilon k_x \pm \sqrt{2\alpha^{3/2}(n+1)}}{2\pi\Delta_L} \quad (88)$$

for f_1 , $J_1 = -\sqrt{\alpha}$, we have

$$\kappa = \frac{-\varepsilon k_x \pm \sqrt{2\alpha^{3/2}n}}{2\pi\Delta_L} \quad (89)$$

Now the energy levels depend on k_x and Landau degeneracy is removed partially.

$$E_{\pm}(k_x, n) = \mu_B B + \hbar v_F \frac{-\varepsilon k_x \pm \sqrt{2\alpha^{3/2}n}}{2\pi\Delta_L} \quad (90)$$

C. External monochromatic electromagnetic wave

Consider an external vector potential

$$A^\mu = A_0^\mu e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \quad (91)$$

which represents a monochromatic plane wave and satisfies Lorentz condition

$$\partial_\mu A^\mu = 0 \quad (92)$$

. $k = \frac{\omega}{c}$. Notice that the plane wave is of speed c , not v_F . Hence

$$\hat{\mathbf{k}} \times \mathbf{E}_0 = c\mathbf{B}_0 \quad (93)$$

then $E_0 \gg v_F B_0$. $k^\mu = (\frac{\omega}{v_F}, \mathbf{k})$, $\mathbf{k} = |\mathbf{k}| \hat{\mathbf{k}}$, $x^\mu = (v_F t, \mathbf{x})$. Note $k^\mu k_\mu = \frac{\omega^2}{v_F^2} - |\mathbf{k}|^2 \neq 0$. This is a major difference between graphene system and real Dirac system in applied electromagnetic wave. We ignore the Zeeman term at first.

$$i\hbar \mathcal{D}\xi = 0 \quad (94)$$

As in [28], denoting $\phi = \mathbf{k} \cdot \mathbf{r} - \omega t$ and $A_\mu = A_\mu(\phi)$. Hence

$$\partial_\mu A^\mu = k^\mu A'_\mu = 0 \quad (95)$$

Or

$$k \cdot A := k_\mu A^\mu = 0 \quad (96)$$

The filed tensor

$$F_{\mu\nu} = k_\mu A'_\nu - k_\nu A'_\mu \quad (97)$$

The second-order equation is

$$(D^\mu D_\mu + \frac{1}{2} \gamma^\mu \gamma^\nu i \frac{q}{\hbar} F_{\mu\nu}) \xi = 0 \quad (98)$$

Using

$$D_\mu D^\mu \xi = (\partial^2 + 2i \frac{q}{\hbar} A^\mu \partial_\mu - \frac{q^2}{\hbar^2} A^2) \xi \quad (99)$$

and

$$[\gamma^\mu, \gamma^\nu] k_\mu A'_\nu = 2(\gamma \cdot k)(\gamma \cdot A') \quad (100)$$

we have

$$\left[\partial^2 + 2i \frac{q}{\hbar} A^\mu \partial_\mu - \frac{q^2}{\hbar^2} A^2 + i \frac{q}{\hbar} (\gamma \cdot k)(\gamma \cdot A') \right] \xi = 0 \quad (101)$$

We seek a solution of this equation, a la the original Volkow ansatz [27][28]

$$\xi = e^{-ip \cdot x} F(\phi) \quad (102)$$

then ($A' = iA$)

$$k^2 F'' - p^2 F - 2ik \cdot p F' + \frac{2q}{\hbar} A \cdot p F - \frac{q^2}{\hbar^2} A^2 F - \frac{q}{\hbar} (\gamma \cdot k)(\gamma \cdot A) F = 0 \quad (103)$$

We can impose (since we can always add to p a multiple of k to meet this condition yet the functional form of ξ remains.) $p^2 = 0$ or some other condition to simplify the equation (103), but it will always be a second-order equation. The reason is that here $k^\mu k_\mu \neq 0$, as distinguishes graphene system from the conventional relativistic electrons.

V. SUMMARY

To summarize, we discussed the Hamiltonian and energy levels of graphene in general constant external electric and magnetic fields. The system is not SO(1,2) Lorentz invariant when Zeeman energy is taken into account. But when the magnetic field is strong enough, SO(1,2) Lorentz invariance is well preserved. Employing the symmetry, we predicted a sequence $\sigma_{xy} = \frac{4e^2}{h} L$ and explain the recently observed Hall conductance $\sigma_{xy} = \frac{4e^2}{h} L$, $L = 0, 4$, which is an indication that the Zeeman levels are not filled at zero temperature. The second-order Dirac equation is derived when the applied field is a monochromatic electromagnetic wave and the difference between graphene system and standard relativistic electrons in this case is revealed.

Acknowledgments

M. M. was partially supported by NSF Grant No. DMR-0804805.

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